

# Two proofs of Størmer's theorem

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## Abstract

The structure of the set of positivity-preserving maps between matrix algebras is notoriously difficult to describe. The notable exceptions are the results by Størmer and Woronowicz from 1960s and 1970s settling the low dimensional cases. By duality, these results are equivalent to the Peres–Horodecki positive partial transpose criterion being able to unambiguously establish whether a state in a  $2 \times 2$  or  $2 \times 3$  quantum system is entangled or separable. However, even in these low dimensional cases, the existing arguments (known to the authors) were based on long and seemingly *ad hoc* computations. We present a simple proof – based on Brouwer's fixed point theorem – for the  $2 \times 2$  case (Størmer's theorem). For completeness, we also include another argument (following the classical outline, but highly streamlined) based on a characterization of extreme self-maps of the Lorentz cone and on a link – noticed by R. Hildebrand – to the  $S$ -lemma, a well-known fact from control theory and quadratic/semi-definite programming.

Denote by  $M_n$  the space of  $n \times n$  complex matrices, by  $M_n^{sa}$  the real-linear subspace of  $n \times n$  Hermitian matrices, and by  $\mathcal{PSD} = \mathcal{PSD}(\mathbb{C}^n)$  the cone of *positive semi-definite* matrices. Further, let  $\mathbf{P} = \mathbf{P}(\mathbb{C}^n)$  denote the cone of *positivity-preserving* maps  $\Phi : M_n^{sa} \rightarrow M_n^{sa}$ , i.e., linear maps verifying  $\Phi(\mathcal{PSD}) \subset \mathcal{PSD}$ .

In this note we will present a short proof of the following 1963 result of Størmer [14].

**Theorem 1** (Størmer's theorem). *A map  $\Phi : M_2^{sa} \rightarrow M_2^{sa}$  belongs to  $\mathbf{P}(\mathbb{C}^2)$  if and only if*

$$\Phi(\rho) = \sum_j A_j \rho A_j^\dagger + \sum_k B_k \rho^T B_k^\dagger \quad (1)$$

for some  $\{A_j, B_k\} \subset M_2$ , where  $\rho^T$  denotes the transpose of  $\rho$ . Moreover, the total number of terms required in (1) does not exceed 4.

In what follows we will describe – for completeness – the background of the result and go over the (rather standard) notation. However, a reader familiar with the subject may just consult the statements of Proposition 2 and Lemma 4, and read the proof of Proposition 2, which together take less than a page. We point that although many proofs of Størmer's theorem appeared in the literature [9, 15, 10, 8], we are not aware of an argument along the ideas of Section 2.

## 1 The background

Since maps of the form

$$\Phi_M(\rho) := M \rho M^\dagger \quad (2)$$

generate the cone of *completely positive* maps, an equivalent restatement of Theorem 1 is that every positivity-preserving map on  $M_2^{sa}$  is *decomposable*, i.e., can be represented as a sum of a completely

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positive map and a *co-completely positive* map (that is, the composition of a completely positive map and the transposition). By duality, Størmer's theorem implies that a state on a  $2 \times 2$  quantum system is separable iff its *partial transpose* is positive [6]. However, these concepts and facts are not needed for our proof; we refer the interested reader to the forthcoming book [1], which also contains a version of the present argument.

The starting point of most proofs of Theorem 1 is the realization that the representation (1) is rather easy to obtain – modulo *very* classical facts – if the map  $\Phi$  is *bistochastic*, that is, *unital* (i.e.,  $\Phi(I) = I$ ) and *trace preserving* (i.e.,  $\text{Tr } \Phi(\rho) = \text{Tr } \rho$  for all  $\rho$  in the domain). This is because the set of states on  $\mathbb{C}^2$

$$D = D(\mathbb{C}^2) = \{\rho \in \mathcal{PSD}(\mathbb{C}^2) : \text{Tr } \rho = 1\}$$

has a particularly simple structure: it is a 3-dimensional *Euclidean ball* (the Bloch ball). More precisely, it is a ball of radius  $1/\sqrt{2}$  in the Frobenius (or Hilbert–Schmidt) norm and centered at  $I/2 =: \rho_*$  (the maximally mixed state). Now,  $\Phi$  being trace- and positivity-preserving is equivalent to  $\Phi(D) \subset D$  and  $\Phi$  being unital is equivalent to  $\Phi(\rho_*) = \rho_*$ , so such  $\Phi$  can be identified with a *linear* operator  $R : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  which maps the unit ball in  $\mathbb{R}^3$  into itself. This means that the norm of  $R$  (the usual, operator or spectral norm) is at most 1 and, consequently,  $R$  is a convex combination of (at most 4, by Carathéodory's theorem applied to the cube  $[-1, 1]^3$ ) isometries of  $\mathbb{R}^3$ , or elements of  $O(3)$ . If  $R \in SO(3)$ , then it is well-known that  $R$  corresponds in the above way to the map that is of the form  $\Phi_U$ , for some  $U \in U(2)$ ; this is an instance of the so-called *spinor map*. Since, as is easy to check, the transpose map  $T$  on  $M_2^{\text{sa}}$  corresponds to a reflection on  $\mathbb{R}^3$ , any  $R \in O(3) \setminus SO(3)$  corresponds to a map of the form  $\Phi_U \circ T$ . Combining these observations we conclude that any bistochastic map  $\Phi : M_2^{\text{sa}} \rightarrow M_2^{\text{sa}}$  can be represented as a convex combination of at most 4 maps that are of the form  $\Phi_U$  or  $\Phi_U \circ T$ , for some  $U \in U(2)$ , which in particular shows that  $\Phi$  verifies the assertion of Theorem 1.

Having settled the bistochastic case, we now want to deduce the general one. Two possible strategies to achieve that are:

1. Focus on maps  $\Phi$  generating *extreme rays* of  $\mathbf{P}(\mathbb{C}^2)$ , and conclude via the Krein–Milman theorem.
2. Focus on maps  $\Phi$  belonging to the *interior* of  $\mathbf{P}(\mathbb{C}^2)$ , and conclude by passing to the closure.

The usual approach, starting with Størmer's proof, was to use the first strategy. We will choose the second one. For a one-stop reading experience, we also present at the end of this note a self-contained proof following the traditional outline, but highly streamlined.

## 2 A proof via Brouwer's theorem

The crucial observation is that maps belonging to the interior of  $\mathbf{P}$  are, in a sense, equivalent to bistochastic ones.

**Proposition 2.** *Let  $\Phi : M_n^{\text{sa}} \rightarrow M_n^{\text{sa}}$  be a linear map which belongs to the interior of  $\mathbf{P}$ , the cone of positivity-preserving maps. Then there exist positive-definite operators  $A, B$  such that*

$$\tilde{\Phi} = \Phi_A \circ \Phi \circ \Phi_B \tag{3}$$

*is bistochastic, where  $\Phi_A, \Phi_B$  are defined by (2).*

Once the Proposition is shown, Theorem 1 readily follows. Indeed, (3) is equivalent to  $\Phi = \Phi_{A^{-1}} \circ \tilde{\Phi} \circ \Phi_{B^{-1}}$ , and appealing to the already proved bistochastic case shows that  $\Phi$  admits a representation of the form (1). Finally, there are no issues with passing to the closure since any term in (1) must belong to the compact set  $\{\Psi \in \mathbf{P} : \Phi - \Psi \in \mathbf{P}\}$ .

Proposition 2 is closely related to Theorem 4.7 from [3]. However, [3] required a constructive – and hence a relatively involved – proof. Similar statements were known earlier for completely positive maps (see, e.g., [2] and its references), but of course that would not be useful for our purposes. We thank David Reeb for bringing these references to our attention.

For our proof of Proposition 2 we will need some notation and two lemmas. First, let  $\Phi^*$  denote the usual functional analytic adjoint of  $\Phi : M_n^{\text{sa}} \rightarrow M_n^{\text{sa}}$ , based on identifying  $M_n^{\text{sa}}$  with its dual via  $\langle \rho, \sigma \rangle_{\text{HS}} := \text{Tr}(\rho\sigma)$ . The following properties of the operation  $*$  are well-known (and easy to show).

**Lemma 3.** *Let  $\Phi : M_n^{\text{sa}} \rightarrow M_n^{\text{sa}}$ . Then*

- (i)  $\Phi \in \mathbf{P}(\mathbb{C}^n)$  if and only if  $\Phi^* \in \mathbf{P}(\mathbb{C}^n)$
- (ii)  $\Phi$  is unital if and only if  $\Phi^*$  is trace-preserving, and vice versa
- (iii) if  $M \in M_n$ , then  $\Phi_M^* = \Phi_{M^\dagger}$ .

The second lemma describes maps belonging to the interior of  $\mathbf{P}(\mathbb{C}^n)$ ; its proof is straightforward and based on very general principles.

**Lemma 4.** *Let  $\Phi : M_n^{\text{sa}} \rightarrow M_n^{\text{sa}}$  be a linear map. The following conditions are equivalent.*

- (i)  $\Phi$  belongs to the interior of  $\mathbf{P}(\mathbb{C}^n)$ .
- (ii)  $\Phi = (1-t)\Psi + t\Omega$  with  $t \in (0, 1]$  and  $\Psi \in \mathbf{P}(\mathbb{C}^n)$ , where  $\Omega(\rho) := (\text{Tr } \rho)\rho_* = (\text{Tr } \rho) \text{ I}/n$  is the completely depolarizing map.
- (iii)  $\Phi^*$  belongs to the interior of  $\mathbf{P}(\mathbb{C}^n)$ .
- (iv) If  $\rho \in D(\mathbb{C}^n)$ , then  $\Phi(\rho)$  is positive definite.
- (v) If  $\rho \in \mathcal{PSD}(\mathbb{C}^n)$  and  $\rho \neq 0$ , then  $\Phi(\rho)$  is positive definite.

*Proof of Proposition 2.* Given positive definite  $A$  and  $B$ , let  $\tilde{\Phi}$  be given by the formula from the Proposition. Then

$$\tilde{\Phi} \text{ is unital} \Leftrightarrow A\Phi(B^2)A = \text{I} \Leftrightarrow \Phi(B^2) = A^{-2} \Leftrightarrow \Phi(B^2)^{-1} = A^2. \quad (4)$$

We next note that, by Lemma 3 (iii), Eq. (3) can be rewritten as  $\tilde{\Phi}^* = \Phi_B \circ \Phi^* \circ \Phi_A$ . Accordingly, by Lemma 3 (ii),

$$\tilde{\Phi} \text{ is trace-preserving} \Leftrightarrow \tilde{\Phi}^* \text{ is unital} \Leftrightarrow B\Phi^*(A^2)B = \text{I} \Leftrightarrow \Phi^*(A^2) = B^{-2}. \quad (5)$$

Solving the last equation in (5) for  $B^2$  and substituting in (4) we are led to a system of equations

$$B^2 = \Phi^*(A^2)^{-1} \text{ and } \Phi(\Phi^*(A^2)^{-1})^{-1} = A^2. \quad (6)$$

The second equation in (6) says that  $S = A^2$  is a fixed point of the function

$$S \mapsto f(S) := \Phi(\Phi^*(S)^{-1})^{-1}. \quad (7)$$

Conversely, if  $S$  is a positive definite fixed point of  $f$ , then  $A = S^{1/2}$  and  $B = \Phi^*(A^2)^{-1/2}$  satisfy (4) and (5) and yield  $\tilde{\Phi}$  which is unital and trace-preserving. Note that, by Lemma 4, the hypothesis “ $\Phi$  belongs to the interior of  $\mathbf{P}$ ” guarantees that all the inverses and negative powers above make sense, and that  $f$  is well-defined and continuous on  $\mathcal{PSD} \setminus \{0\}$ .

To find a fixed point of  $f$  we want to use Brouwer’s fixed-point theorem, which requires a (continuous) function that is a self-map of a compact convex set. One way to arrive at that setting is to consider  $f_1 : D(\mathbb{C}^n) \rightarrow D(\mathbb{C}^n)$  defined by

$$f_1(\sigma) = \frac{f(\sigma)}{\text{Tr } f(\sigma)}. \quad (8)$$

It then follows that there is  $\sigma_0 \in D(\mathbb{C}^n)$  such that  $f_1(\sigma_0) = \sigma_0$  and hence  $f(\sigma_0) = \alpha\sigma_0$ , where  $\alpha = \text{Tr } f(\sigma_0) > 0$ . The final step is to note that if we choose – as before –  $A = \sigma_0^{1/2}$  and  $B = \Phi^*(A^2)^{-1/2}$ , then the corresponding  $\tilde{\Phi}$  is trace-preserving and satisfies  $\tilde{\Phi}(\text{I}) = \alpha^{-1}\text{I}$ , which is only possible if  $\alpha = 1$ . In other words,  $\sigma_0$  is a fixed point of  $f$  that we needed to conclude the argument.  $\square$

*Remark 5.* If properly stated, Proposition 2 generalizes – with essentially the same proof – to maps  $\Phi : M_m^{\text{sa}} \rightarrow M_n^{\text{sa}}$  with  $m \neq n$ . The correct conditions are that  $\tilde{\Phi}$  is trace preserving and that it sends the maximally mixed state  $\text{I}/m \in D(\mathbb{C}^m)$  to the maximally mixed state  $\text{I}/n \in D(\mathbb{C}^n)$ . This suggests in particular a possible path to a simple proof of Woronowicz’s theorem [16] (a version of Theorem 1 for maps  $\Phi : M_2^{\text{sa}} \rightarrow M_3^{\text{sa}}$ ) by reducing it to the case of maps verifying these two conditions.

### 3 A traditional proof

The second proof we present is based on the more traditional strategy, a description of the maps  $\Phi$  generating *extreme rays* of  $\mathbf{P}(\mathbb{C}^2)$ . That description is most conveniently expressed as a statement about the Lorenz cone

$$\mathcal{L}_m = \{x = (x_0, x_1, \dots, x_{m-1}) : x_0 \geq 0, q(x) \geq 0\},$$

where  $q(x) := x_0^2 - \sum_{k=1}^{m-1} x_k^2$ . If  $\mathbf{P}(\mathcal{L}_m)$  is the cone of linear maps on  $\mathbb{R}^m$  that preserve  $\mathcal{L}_m$ , we have [11]

**Proposition 6.** *Let  $\Phi : \mathbb{R}^m \rightarrow \mathbb{R}^m$  be a linear map which generates an extreme ray of  $\mathbf{P}(\mathcal{L}_m)$ . Then either  $\Phi$  is an automorphism of  $\mathcal{L}_m$  or  $\Phi$  is of rank one, in which case  $\Phi = |u\rangle\langle v|$  for some  $u, v \in \partial\mathcal{L}_m \setminus \{0\}$ . If  $m > 2$ , the converse implication also holds.*

Since  $\mathcal{PSD}(\mathbb{C}^2)$  is isomorphic to  $\mathcal{L}_4$ , Proposition 6 yields a characterization of extreme rays of  $\mathbf{P}(\mathbb{C}^2)$  and – by the Krein–Milman theorem – reduces the proof of Theorem 1 to showing that the corresponding extreme maps admit a representation of type (1).

To establish the last fact, we note that the structure of the set of automorphisms of  $\mathcal{L}_m$  is very well understood: they are of the form  $t\Phi$ , where  $t > 0$  and  $\Phi \in \mathbf{O}^+(1, m-1)$ , the *orthochronous subgroup* of the *Lorentz group*  $\mathbf{O}(1, m-1)$  of transformations preserving the quadratic form  $q(x) = x_0^2 - \sum_{k=1}^{m-1} x_k^2$ . However, for  $m = 4$  and for our purposes, it is more convenient to use the fact that automorphisms of  $\mathcal{PSD}(\mathbb{C}^n)$  are of the form  $\rho \mapsto V\rho V^\dagger$  or  $\rho \mapsto V\rho^T V^\dagger$  for some  $V \in \mathbf{GL}(n)$ , which immediately yields a representation of type (1). (This is an instance of Kadison’s theorem [7], which for  $n = 2$  is elementary and very simple.)

The case of rank one maps is even simpler: every element of  $\partial\mathcal{PSD}(\mathbb{C}^2) \setminus \{0\}$  is of the form  $|\varphi\rangle\langle\varphi|$ ,  $\varphi \in \mathbb{C}^2 \setminus \{0\}$ , and so  $\Phi$  can be represented as

$$\Phi(\rho) = \text{Tr}(\rho|\xi\rangle\langle\xi|)|\psi\rangle\langle\psi| = |\psi\rangle\langle\xi|\rho|\xi\rangle\langle\psi|.$$

In other words,  $\Phi = \Phi_{|\psi\rangle\langle\xi|}$ , as needed. It should be noted, however, that – in absence further refinement – this argument involves later an application of Carathéodory’s theorem in a 15-dimensional space (say, in  $\{\Phi : \text{Tr}(\Phi(I)) = n\}$ ), leading to a bound of 16 on the number of terms in (1).

The above scheme of the proof of Størmer’s theorem was apparently folklore for some time; it appears explicitly in [12]. However, its value was limited by the fact that the proof of Proposition 6 given in [11] was itself long and computational. Our contribution, if any, consists in streamlining of the argument given in [4, 5], which rediscovered Proposition 6 and noted its relevance to the entanglement theory. The proof is based on the so-called *S-lemma* [17], a well-known fact from control theory and quadratic/semi-definite programming.

**Lemma 7** (*S-lemma*). *Let  $F, G$  be  $n \times n$  symmetric real matrices. Assume that there is an  $\bar{x} \in \mathbb{R}^n$  such that  $\langle \bar{x} | G | \bar{x} \rangle > 0$ . Then the following two statements about such  $F, G$  are equivalent:*

- (i) *if  $x \in \mathbb{R}^n$  verifies  $\langle x | G | x \rangle \geq 0$ , then  $\langle x | F | x \rangle \geq 0$*
- (ii) *there exists  $\mu \geq 0$  such that  $F - \mu G$  is positive semi-definite.*

We postpone the proof of the Lemma until the end of this Appendix and show how it implies the Proposition. (We leave out the “converse” part, which is easier and not needed for our purposes.)

*Proof of Proposition 6.* The case  $\text{rank } \Phi = 1$  is an immediate consequence of the following elementary observation, which completely characterizes extreme rays generated by rank one maps in a very general setting (we only need the “only if” part, which is very easy).

**Lemma 8.** *Let  $\mathcal{C} \subset \mathbb{R}^n$  be a nondegenerate cone and let  $\mathbf{P}(\mathcal{C})$  be the cone of linear maps preserving  $\mathcal{C}$ . A rank one map  $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  generates an extreme ray of  $\mathbf{P}(\mathcal{C})$  iff it is of the form  $\Phi = |u\rangle\langle v|$ , with  $u$  and  $v$  generating extreme rays of respectively  $\mathcal{C}$  and the dual cone  $\mathcal{C}^*$ .*

Above,  $\mathcal{C}$  being *nondegenerate* means that  $\dim \mathcal{C} = n$  and  $-\mathcal{C} \cap \mathcal{C} = \{0\}$ , while the *dual cone* is defined by  $\mathcal{C}^* := \{x \in \mathbb{R}^n : \langle x|y \rangle \geq 0 \text{ for all } y \in \mathcal{C}\}$ .

Next, assume that  $\text{rank } \Phi \geq 2$ . Let  $J \in \text{O}(n)$  be the diagonal matrix with diagonal entries  $1, -1, \dots, -1$ ; then  $\langle x|J|x \rangle = x_0^2 - \sum_{k=1}^{m-1} x_k^2 = q(x)$  for  $x \in \mathbb{R}^n$ . The map  $\Phi$  preserving  $\mathcal{L}_n$  (and hence  $-\mathcal{L}_n$ ) means that the hypothesis (i) of Lemma 7 is satisfied with  $G = J$  and  $F = \Phi^* J \Phi$ . Since clearly  $-J$  is not positive definite, it follows that there is  $\mu \geq 0$  and a positive semi-definite operator  $Q$  such that

$$\Phi^* J \Phi = \mu J + Q. \quad (9)$$

We now notice that since  $\text{rank } \Phi \geq 2$ , there is  $y = \Phi x \neq 0$  such that  $y_0 = 0$ . In particular,  $\langle x|\Phi^* J \Phi|x \rangle = \langle y|J|y \rangle < 0$ . Given that  $\langle x|Q|x \rangle \geq 0$ , it follows that  $\mu$  can not be 0. Next, if  $Q = 0$ , (9) means precisely that  $\mu^{1/2} \Phi \in \text{O}(1, n-1)$  and so  $\Phi$  is an automorphism of  $\mathcal{L}_n$ .

To complete the argument, we will show that if  $Q \neq 0$ , then there is a rank one operator  $\Delta$  such that  $\Phi \pm \Delta \in \mathbf{P}(\mathcal{L}_m)$ . Since  $\Phi$  and  $\Delta$  have different ranks, they are not proportional. Hence  $\Phi + \Delta$  and  $\Phi - \Delta$  do not belong to the ray generated by  $\Phi$ , which implies that the ray is not extreme.

Let  $|v\rangle\langle v|$ ,  $v \neq 0$ , be one of the terms appearing in the spectral decomposition of  $Q$ ; then  $Q = Q' + |v\rangle\langle v|$ , where  $Q'$  is positive semi-definite. Next, let  $u \in \mathbb{R}^n \setminus \{0\}$  be such that  $\Phi^* J u = \delta v$ , where  $\delta$  is either 1 or 0. Such  $u$  exists: if  $\Phi^*$  is invertible, then  $u = J(\Phi^*)^{-1}v$  satisfies  $\Phi^* J u = v$ , while in the opposite case the nullspace of  $\Phi^* J$  is nontrivial. We will show that, for some  $\varepsilon > 0$ ,

$$\Phi + s|u\rangle\langle v| \in \mathbf{P}(\mathcal{L}_m) \text{ if } |s| \leq \varepsilon, \quad (10)$$

thus supplying the needed  $\Delta = \varepsilon|u\rangle\langle v|$ . We have, by (9) and by the choice of  $u$ ,

$$\begin{aligned} (\Phi + s|u\rangle\langle v|)^* J (\Phi + s|u\rangle\langle v|) &= \mu J + Q + 2s\delta|v\rangle\langle v| + s^2|v\rangle\langle u|J|u\rangle\langle v| \\ &= \mu J + Q' + (1 + 2s\delta + s^2\langle u|J|u\rangle)|v\rangle\langle v|. \end{aligned} \quad (11)$$

Since clearly  $1 + 2s\delta + s^2\langle u|J|u\rangle \geq 0$  if  $|s|$  is sufficiently small, it follows that, for such  $s$ ,  $(\Phi + s|u\rangle\langle v|)^* J (\Phi + s|u\rangle\langle v|) - \mu J$  is positive semidefinite. Thus we can deduce from the easy part of Lemma 7 that  $\Phi + s|u\rangle\langle v| \in \mathbf{P}(\mathcal{L}_m)$ , as needed. (To be precise, we need to exclude the possibility that  $\Phi + s|u\rangle\langle v| \in -\mathbf{P}(\mathcal{L}_m)$ , but this is simple.)  $\square$

It remains to prove Lemma 7. We follow [13]; we first restate the Lemma in a simpler form [18].

**Lemma 9** (*S-lemma reformulated*). *Let  $M, N$  be  $n \times n$  symmetric real matrices. The following two statements are equivalent:*

- (i)  $\{x \in \mathbb{R}^n : \langle x|M|x \rangle \geq 0\} \cup \{x \in \mathbb{R}^n : \langle x|N|x \rangle \geq 0\} = \mathbb{R}^n$
- (ii) *there exists  $t \in [0, 1]$  such that the matrix  $(1-t)M + tN$  is positive semi-definite.*

Lemma 7 is an easy consequence of Lemma 9 applied with  $M = F$  and  $N = -G$ .

*Proof of Lemma 9.* The implication (ii)  $\Rightarrow$  (i) is straightforward. To show that (i)  $\Rightarrow$  (ii), we argue by contradiction. Denote  $M_t = (1-t)M + tN$  and assume that, for every  $t \in [0, 1]$ , the smallest eigenvalue  $\lambda_t$  of  $M_t$  is strictly negative. Note that  $t \mapsto \lambda_t$  is continuous. For  $t \in [0, 1]$ , set

$$\Lambda_t := \{x \in S^{n-1} : M_t x = \lambda_t x\} \neq \emptyset.$$

Then  $t \mapsto \Lambda_t$  upper semicontinuous in the sense that  $t_n \rightarrow t$ ,  $x_n \in \Lambda_{t_n}$  and  $x_n \rightarrow x$  imply  $x \in \Lambda_t$ .

Consider the sets  $A = \{x \in \mathbb{R}^n : \langle x|M|x \rangle \geq 0\}$  and  $B = \{x \in \mathbb{R}^n : \langle x|N|x \rangle \geq 0\}$ . We have  $A \cup B = \mathbb{R}^n$  by hypothesis. Since  $M_0 = M$ , it follows that  $\Lambda_0 \cap A = \emptyset$  and so  $\Lambda_0 \subset B$ . Similarly,  $\Lambda_1 \subset A$ . Set

$$\tau = \sup\{t \in [0, 1] : \Lambda_t \cap B \neq \emptyset\}.$$

We now note that  $\Lambda_\tau \cap B \neq \emptyset$ ; this is immediate if  $\tau = 0$  and follows from upper semicontinuity of  $t \mapsto \Lambda_t$  if  $\tau > 0$ . For essentially the same reasons,  $\Lambda_\tau \cap A \neq \emptyset$ .

We now claim that  $\Lambda_\tau \cap A \cap B \neq \emptyset$ . This is clear if the eigenvalue  $\lambda_\tau$  is simple (note that all three sets,  $\Lambda_\tau$ ,  $A$  and  $B$ , are symmetric by definition). On the other hand, if the multiplicity of  $\lambda_t$  equals  $k > 1$ , then  $\Lambda_\tau$  is a  $(k - 1)$ -dimensional sphere and hence is connected. Consequently, the closed nonempty sets  $\Lambda_\tau \cap A$  and  $\Lambda_\tau \cap B$ , the union of which is  $\Lambda_\tau$ , must have a nonempty intersection.

To conclude the argument, choose  $x \in \Lambda_\tau \cap A \cap B \neq \emptyset$ . Then, since  $x \in \Lambda_\tau$ ,

$$\langle x | M_\tau | x \rangle = \lambda_t < 0.$$

On the other hand, since  $x \in A \cap B$ ,

$$\langle x | M_\tau | x \rangle = (1 - \tau) \langle x | M | x \rangle + \tau \langle x | N | x \rangle \geq 0,$$

a contradiction. □

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